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Abstract
First a set of coherent states *à la* Klauder is formally constructed for the Coulomb problem in a curved space of constant positive curvature. Then the flat-space limit is taken to reduce the set for the radial Coulomb problem to a set of hydrogen atom coherent states corresponding to both the discrete and the continuous portions of the spectrum for a fixed ℓ sector.

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1. Introduction

In [1], Klauder proposed a set of coherent states in relation with the bound state portion of the hydrogen atom, generalizing the harmonic oscillator coherent states so as to preserve the following three properties; they are (i) continuous in their parameters, (ii) admit a resolution of unity and (iii) are temporally stable (i.e., evolve among themselves in time). In fact there are a number of ways to generalize the harmonic oscillator coherent states [2, 3]. Most generalizations, notably Perelomov's, are based on group structures. Klauder stipulates coherent states without resort to a group.

The proposed coherent states with an energy spectrum $E_n = \omega e_n (n = 0, 1, 2, \ldots; e_0 = 0)$ are labelled by two real parameters s $(0 \leqslant s < \infty)$ and γ $(-\infty < \gamma < \infty)$ as

$$|s, \gamma\rangle = M(s^2) \sum_{n=0}^{\infty} \frac{s^n \, \mathrm{e}^{-\mathrm{i}\gamma e_n}}{\sqrt{\rho_n}} |n\rangle \tag{1}$$

where $|n\rangle$ is the eigenstate belonging to E_n and ρ_n is the nth moment of a probability distribution function $\rho(u) > 0$,

$$\rho_n = \int_0^{\bar{u}} u^n \rho(u) \, \mathrm{d}u. \tag{2}$$

For the harmonic oscillator, $\rho(u) = e^{-u}$ leads to the desirable result $\rho_n = n!$. However, in general, the coherent states (1) as proposed by Klauder have ambiguity in the choice of $\rho(u)$. The normalization factor $M(s^2)$ is determined so as to satisfy $\langle s, \gamma | s, \gamma \rangle = 1$; namely,

$$M(s^2)^{-2} = \sum_{n=0}^{\infty} \frac{s^{2n}}{\rho_n}. \tag{3}$$

With the Hamiltonian $\hat{H}$ such that $\hat{H}|n\rangle = \omega e_n |n\rangle$, it is apparent that

$$e^{-i\hat{H}t}|s, \gamma\rangle = |s, \gamma + \omega t\rangle \tag{4}$$

which is taken in [1] as the exhibition of temporal stability of the coherent states. The states satisfy the resolution of unity,

$$\int d\mu(s, \gamma)|s, \gamma\rangle\langle s, \gamma| = \hat{1}_{\text{dis}} \tag{5}$$

with a measure $\mu(s, \gamma)$ defined by

$$\int d\mu(s, \gamma) f(s, \gamma) = \lim_{\Gamma \to \infty} \frac{1}{2\Gamma} \int_0^{\infty} k(s^2)\, ds^2 \int_{-\Gamma}^{\Gamma} d\gamma\, f(s, \gamma) \tag{6}$$

provided that

$$\lim_{\Gamma \to \infty} \frac{1}{2\Gamma} \int_{-\Gamma}^{\Gamma} d\gamma\, e^{i\gamma(e_n - e_{n'})} = \delta_{n,n'}, \tag{7}$$

that is, that all e_n are distinct (no degeneracies). In (6),

$$k(s^2) = \rho(s^2)/M(s^2)^2, \tag{8}$$

which remains unspecified until the form of $\rho(s^2)$ in (2) is given. Gazeau and Klauder [4], letting $s^2 = J$, imposed an additional condition, called the action identity [4],

$$\langle J, \gamma | \hat{H} | J, \gamma \rangle = \omega J, \tag{9}$$

which leads ρ_n to the form,

$$\rho_n = \prod_{j=1}^{n} e_j, \qquad \rho_0 = 1. \tag{10}$$

This condition suggests one to interpret the parameter J as the classical action variable conjugate to the angle variable γ. A remark will be made in section 4 concerning a possible use of J for the semiclassical quantization condition.

Gazeau and Klauder [4] also proposed coherent states for continuum dynamics. For a Hamiltonian with a non-degenerate continuous spectrum $0 < \omega\varepsilon < \omega\bar{\varepsilon}$, the proposed coherent states take the form,

$$|s, \gamma\rangle = M(s^2) \int_0^{\bar{\varepsilon}} \frac{s^{\varepsilon} e^{-i\gamma\varepsilon}}{\sqrt{\rho(\varepsilon)}} |\varepsilon\rangle\, d\varepsilon, \tag{11}$$

where

$$M(s^2)^{-2} = \int_0^{\bar{\varepsilon}} \frac{s^{2\varepsilon}}{\rho(\varepsilon)}\, d\varepsilon \tag{12}$$

to meet $\langle s, \gamma | s, \gamma \rangle = 1$ for $0 \leqslant s < \bar{s}$. The function $\rho(\varepsilon)$ in (11) is determined with an appropriate non-negative weighting function $\sigma(s) \geqslant 0$ as

$$\rho(\varepsilon) = \int_0^{\bar{\varepsilon}} s^{2\varepsilon} \sigma(s)\, ds. \tag{13}$$

These coherent states for a continuous spectrum evolve in time among themselves. With $d\mu(s, \gamma) = (1/2\pi)M(s)^{-2}\sigma(s)\,ds\,d\gamma$, the resolution of unity,

$$\int d\mu(s, \gamma)|s, \gamma\rangle\langle s, \gamma| = \hat{1}_{cont},\tag{14}$$

is fulfilled. In [4], the resolution of unity is set up independently for the discrete and the continuous case.

In this paper, we first consider within the Gazeau–Klauder framework a set of coherent states for the radial Coulomb problem in a curved space of constant positive curvature. Then, taking the flat-space limit, we obtain a set of coherent states for the continuous part as well as the discrete portion of the hydrogen spectrum in a unified manner. Although the spectrum of the Coulomb system in the curved space is wholly discrete, the set of coherent states we derive in the flat-space limit consists of the discrete and continuous portions. In particular, for the S-waves, the set coincides with that constructed by Gazeau and Klauder for the hydrogen-like spectrum [4] if the continuous portion is ignored.

2. The Coulomb problem in a uniformly curved space

Schrödinger [5] was the first to find quantum mechanical solutions for the Coulomb problem in a curved space of constant positive curvature. He considered this problem as an example that can be solved by the factorization procedure but is not tractable by other methods (see also [6]). Soon after, however, Stevenson [7] succeeded in obtaining the solutions by a conventional manner. Indeed there are various ways to approach the problem. It may be worth mentioning that the same problem has been solved by the dynamical group approach [8] and by the path integral approach [9]. It is interesting that the system has only a discrete spectrum unlike the usual hydrogen atom in flat space and may be seen as a compactified version of the usual Coulomb problem. It is natural to expect that the discrete spectrum of the system will generate the entire spectrum of the hydrogen atom including both the continuous and the discrete portions when the curvature of the space diminishes. We shall explore this limiting property later. First we wish to construct the coherent states à la Klauder for the radial part of the Coulomb system.

We assume that space is uniformly curved with a positive curvature $K = 1/R^2 > 0$. Then the curved space may be realized as a three-dimensional sphere (S^3) of radius R imbedded in a four-dimensional Euclidean space. The line element ds of the space is given in polar coordinates by

$$ds^2 = \frac{dr^2}{1 - r^2/R^2} + r^2(d\theta^2 + \sin^2\theta\,d\phi^2).\tag{15}$$

Or, with $\sin\chi = r/R\,(\chi \in [0, \pi])$, it can be put in the form,

$$ds^2 = R^2\,d\chi^2 + R^2\sin^2\chi\,(d\theta^2 + \sin^2\theta\,d\phi^2).\tag{16}$$

The Coulomb potential on the sphere [5] is

$$V(\chi) = -\frac{Ze^2}{R}\cot\chi\tag{17}$$

which satisfies the harmonic condition [6],

$$\frac{d}{d\chi}\left(\sin^2\chi\frac{dV}{d\chi}\right) = 0.$$

The Hamiltonian operator for this Coulomb system with a unit mass is given by

$$\hat{H} = -\frac{1}{2}\hat{\Delta} - \frac{Ze^2}{R}\cot\chi,\tag{18}$$

where $\hbar = 1$ and $\hat{\Delta}$ is the Laplace–Beltrami operator of $SO(4)$. The corresponding Schrödinger equation may be expressed as

$$\left[\frac{1}{\sin^2 \chi} \frac{\partial}{\partial \chi} \left(\sin^2 \chi \frac{\partial}{\partial \chi} \right) - \frac{\hat{L}^2}{\sin^2 \chi} + 2\sqrt{2\omega} R \cot \chi + 2R^2 E \right] \psi(\chi, \theta, \phi) = 0 \tag{19}$$

where $\omega = Z^2 e^4 / 2$, and $\hat{L}^2$ is the Casimir invariant of $SO(3)$,

$$\hat{L}^2 = -\left[\frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial}{\partial \theta} \right) + \frac{1}{\sin^2 \theta} \frac{\partial^2}{\partial \phi^2} \right]. \tag{20}$$

It is obvious that the $SO(3)$ portion can be separated by letting the wavefunction be the product of the radial function and the spherical harmonics, $\psi(\chi, \theta, \phi) \sim w_\ell(\chi) Y_\ell^m(\theta, \phi)$. The radial function $w_\ell(\chi)$ obeys

$$(\hat{H}_\ell - E) w_\ell(\chi) = 0 = \tag{21}$$

with the radial Hamiltonian,

$$\hat{H}_\ell = -\frac{1}{2R^2} \left[\frac{\partial^2}{\partial \chi^2} + 2 \cot \chi \frac{\partial}{\partial \chi} - \frac{\ell(\ell+1)}{\sin^2 \chi} + 2\sqrt{2\omega} R \cot \chi \right]. \tag{22}$$

From this follows the degenerate energy spectrum [5–7]

$$E_N = \frac{N^2 - 1}{2R^2} - \frac{\omega}{N^2}, \qquad (N = 1, 2, 3, \ldots), \tag{23}$$

and the corresponding eigenfunctions [6, 7, 9]

$$w_{N,\ell}(\chi) \sim \sin^\ell \chi \, e^{-i\chi(N-\ell-1+i\lambda_n)} {}_2F_1(\ell - N + 1, \ell + 1 - i\lambda_n; 2\ell + 2; 1 - e^{2i\chi}). \tag{24}$$

In the above, ${}_2F_1(\alpha, \beta; \gamma; z)$ is Gauss's hypergeometric function, and

$$\lambda_n = -\frac{R}{a(n + \ell + 1)} = -\frac{R}{aN}, \tag{25}$$

with $a = (2\omega)^{-1/2} = (Ze^2)^{-1}$.

At this point we note that the present polar coordinate realization of the system, even being in a curved space, is not degeneracy free. Since an analogue of the Runge–Lentz vector exists and commutes with the Hamiltonian, the accidental degeneracy persists [10, 11]. Therefore, we consider only the coherent states associated with the radial wavefunctions. Fixing ℓ we label the wavefunctions by the radial quantum number $n = 0, 1, 2, \ldots$ rather than the principal quantum number $N = n + \ell + 1 = 1, 2, 3, \ldots$.

With the radial quantum number n, the energy spectrum (23) and the wavefunctions (24) may be given, respectively, by

$$E_n = \frac{(n + \ell)(n + \ell + 2)}{2R^2} - \frac{\omega}{(n + \ell + 1)^2}, \qquad (n = 0, 1, 2, \ldots), \tag{26}$$

and

$$w_{n,\ell}(\chi) = C_{n,\ell} \sin^\ell \chi \, e^{-i\chi(n+i\lambda_n)} {}_2F_1(-n, \ell + 1 - i\lambda_n; 2\ell + 2; 1 - e^{2i\chi}) \tag{27}$$

with the normalization factor [9]

$$C_{n,\ell} = e^{i\pi(2n+\ell+1)/2} \frac{2^{\ell+1}}{\Gamma(2\ell+2)} \left[\frac{i\{(n+\ell+1)^2 + \lambda_n^2\} \Gamma(\ell+1+i\lambda_n) \Gamma(n+2\ell+2)}{R^3 \kappa_n \Gamma(i\lambda_n - \ell) \Gamma(n+1)} \right]^{1/2} \tag{28}$$

where

$$\kappa_n = \min\{|n + \ell + 1|, |\lambda_n|\}. \tag{29}$$

Now we write down the coherent states *à la* Klauder for the radial Coulomb problem on the sphere; namely,

$$|s, \gamma\rangle = \mathcal{M}(s^2) \sum_{n=0}^{\infty} \frac{s^n \, e^{-i\gamma[n]_R}}{\sqrt{[n]_R!}} |n\rangle. \tag{30}$$

Here we note that $|n\rangle$ is the nth energy eigenvector so that $w_{n,\ell}(\chi) = \langle\chi|n\rangle$. Also we have defined a generalized number $[n]$ by

$$[n] = (E_n - E_0)/\omega \quad (n = 0, 1, 2, \ldots). \tag{31}$$

and $[0]! = 1$. The subscript R stands for a finite radius R of curvature. For (26), we have

$$[n]_R! = \prod_{m=1}^{n} [m]_R = \prod_{m=1}^{n} \left[\frac{m(m + 2\ell + 2)}{(m + \ell + 1)^2(\ell + 1)^2} \left(1 + \frac{(m + \ell + 1)^2(\ell + 1)^2}{2\omega R^2} \right) \right] \tag{32}$$

and

$$\mathcal{M}(s^2)^{-2} = \sum_{n=0}^{\infty} \frac{s^{2n}}{[n]_R!}. \tag{33}$$

Since $[n]_R!$ and hence $\mathcal{M}(s^2)$ cannot be given in closed form, the coherent states (30) so constructed for the radial Coulomb problem in curved space are not particularly interesting until their flat-space limits are taken. Nevertheless, it is obvious that the set of coherent states given above possess all the properties (i)–(iii) of Klauder's coherent states [1] plus the action identity of Gazeau and Klauder [4].

3. The coherent states for the radial Coulomb problem in flat space

Next we consider the flat-space limits where the radius of curvature R tends to infinity. We conjecture that the Coulomb problem on the sphere with $Z = 1$ goes over to the hydrogen atom problem (with $m_e = 1$) in the flat-space limit. By doing so, we expect that the discrete energy spectrum of the Coulomb problem on the sphere will correspond to both the discrete and the continuous parts of the hydrogen atom spectrum in flat space [9].

Before taking the limit, we introduce the critical number n_c for which the energy becomes zero, that is, $E_{n_c} = 0$, or

$$(n_c + \ell)(n_c + \ell + 2)(n_c + \ell + 1)^2 = 2\omega R^2. \tag{34}$$

Then we separate spectrum (26) into two parts: (a) $E_n < 0$ and (b) $E_n \geqslant 0$, and consider their limiting cases separately.

Case (a). $E < 0 \, (n < n_c)$. It is apparent from (34) that as R approaches infinity n_c goes to infinity as fast as $\sqrt{R}$. Accordingly the first term of the energy spectrum (26) for $n < n_c$ tends to zero as

$$\left| E_n + \frac{\omega}{(n + \ell + 1)^2} \right| = \frac{(n + \ell)(n + \ell + 2)}{2mR^2} < \frac{(n_c + \ell)(n_c + \ell + 2)}{2mR^2} \sim \frac{1}{R} \to 0. \tag{35}$$

Since $n_c \to \infty$, the energy spectrum bounded above by zero takes the form,

$$E_n = -\frac{\omega}{(n + \ell + 1)^2} \quad (n = 0, 1, 2, \ldots) \tag{36}$$

which coincides, as is expected, with the discrete hydrogen atom spectrum.

Now note that Gauss's hypergeometric function is reduced to Kummer's confluent hypergeometric function by the limiting procedure,

$$\lim_{\beta \to \infty} {}_2F_1(\alpha, \beta; \gamma; z/\beta) = {}_1F_1(\alpha; \gamma; z)$$

and that for large $|z|$

$$\frac{\Gamma(z + \ell + 1)}{\Gamma(z - l)} \sim z^{2\ell+1}.$$

Recalling also that $\lambda_n = -R/\{a(n+\ell+1)\}$, we obtain the following limiting values for $E < 0$:

$$\lim_{R \to \infty} {}_2F_1(-n - 1, \ell + 1 - i\lambda_n; 2\ell + 2; 1 - e^{2i\chi}) = {}_1F_1(-n - 1; 2\ell + 2; 2r/a(n + \ell + 1))$$

$$\lim_{R \to \infty} \exp[-i\chi(n + 1 + i\lambda_n)] = \exp[-r/a(n + \ell + 1)]$$

$$\lim_{R \to \infty} \sin^\ell \chi \left[\frac{i\{(n + \ell + 1)^2 + \lambda_n^2\}\Gamma(\ell + 1 + i\lambda_n)}{R^3(n + \ell + 1)\Gamma(i\lambda_n - \ell)} \right]^{1/2} = \left[\frac{ir}{a(n + \ell + 1)} \right]^\ell \left[\frac{1}{a^3(n + \ell + 1)^4} \right]^{1/2}. \tag{37}$$

Thus, in the flat-space limit, the radial function (24) takes the form,

$$u_{n,\ell}(r) = C_n \left[\frac{2r}{a(n + \ell + 1)} \right]^\ell e^{-r/a(n+\ell+1)} {}_1F_1(-n - 1; 2\ell + 2; 2r/a(n + \ell + 1)) \tag{38}$$

with the normalization constant,

$$C_n = \frac{1}{(2\ell + 1)!} \left[\left(\frac{2}{a(n + \ell + 1)} \right)^3 \frac{(n + 2\ell + 1)!}{2(n + \ell + 1)(n + 1)!} \right]^{1/2}. \tag{39}$$

This result is in fact the normalized hydrogen atom radial wavefunction in units where $\hbar = 1$. See, e.g., [12].

Case (b). $E \geqslant 0$ ($n \geqslant n_c$). For large R, we approximate $\Delta n/R$ for $n > n_c$ by dk with a continuous parameter $k > 0$, so that by integration

$$n - n_c = kR. \tag{40}$$

In the limit $R \to \infty$, the energy spectrum behaves as

$$E_n = \frac{(kR + n_c + \ell)(kR + n_c + \ell + 2)}{2R^2} - \frac{\omega}{(kR + n_c + \ell + 1)^2} \to \frac{k^2}{2}. \tag{41}$$

As a result, the discrete spectrum (26) for $E \geqslant 0$ turns into a continuous spectrum,

$$E(k) = \frac{k^2}{2} \quad (0 \leqslant k). \tag{42}$$

In this continuous case, for large R, we must replace λ_n by $-1/ak$. In a way similar to evaluating the discrete limits (37), we calculate the continuous limiting values,

$$\lim_{R \to \infty} {}_2F_1(\ell - n, \ell + 1 - i\lambda_n; 2\ell + 2; 1 - e^{2i\chi}) = {}_1F_1(\ell + 1 + i/ak; 2\ell + 2; 2ikr)$$

$$\lim_{R \to \infty} \exp[-i\chi(n - \ell + i\lambda_n)] = \exp[-ikr]$$

$$\lim_{R \to \infty} \sin^\ell \chi \left[\frac{i}{R^3} \frac{((n + 1)^2 + \lambda_n^2)\Gamma(n + \ell + 2)}{|\lambda_n|\Gamma(n - \ell + 1)} \right]^{1/2} = 2^{-\ell} e^{i\pi/4} \sqrt{a} k^2 (2kr)^\ell. \tag{43}$$

Using these results, we arrive at the limiting wavefunction belonging to the continuous spectrum for $E \geqslant 0$,

$$v_{k,\ell}(r) = \left(\frac{2a}{\pi}\right)^{1/2} \frac{k^2(2kr)^\ell}{(2\ell+1)!} |\Gamma(\ell+1-\mathrm{i}/ak)| \sinh^{1/2}(\pi/ak)$$
$$\times \mathrm{e}_1^{\mathrm{i}kr} F_1(\ell+1+\mathrm{i}/ak; 2\ell+2; 2\mathrm{i}kr). \tag{44}$$

In this manner, we see that the discrete dynamics of the Coulomb system in curved space leads to the discrete and continuous regimes of the hydrogen atom in flat space.

Now we consider the flat-space limit of the coherent states (30) for fixed ℓ:

$$|s, \gamma\rangle = \lim_{R\to\infty} \mathcal{M}(s^2) \left[\sum_{E<0} + \sum_{E\geqslant 0}\right] \frac{s^n \, \mathrm{e}^{-\mathrm{i}\gamma[n]}}{\sqrt{[n]_R!}} |n\rangle. \tag{45}$$

Corresponding to the limiting discrete spectrum (36) for $E < 0$, we write

$$\lim_{R\to\infty} [n]_R = [n] \tag{46}$$

and have

$$[n]! = \prod_{m=1}^{n} \frac{m(m+2\ell+2)}{(\ell+1)^2(m+\ell+1)^2} = \frac{n!}{(\ell+1)^{2n}} \frac{(2\ell+3)_n}{[(\ell+2)_n]^2}, \tag{47}$$

where $(z)_n$ is the Pochhammer symbol,

$$(z)_n = \Gamma(z+n)/\Gamma(z), \qquad (z)_0 = 1. \tag{48}$$

In particular, for the S-wave case ($\ell = 0$), this reduces to the result given for the Coulomb-like spectrum in [4]:

$$\rho_n = \frac{n+2}{2(n+1)}. \tag{49}$$

With (47) the discrete portion of the coherent states becomes

$$|s, \gamma\rangle_{\mathrm{disc}} = \mathcal{N}(s^2) \sum_{n=0}^{\infty} \frac{s^n \, \mathrm{e}^{-\mathrm{i}\gamma[n]}}{\sqrt{[n]!}} |n\rangle_{\mathrm{disc}}, \tag{50}$$

where

$$\mathcal{N}(s^2) = \lim_{R\to\infty} \mathcal{M}(s^2) \tag{51}$$

which will be evaluated shortly. The discrete eigenstates $|n\rangle$ and the radial wavefunctions (38) with a fixed ℓ are related by $u_{n,\ell}(r) = \langle r|n\rangle$. Naturally result (50) for the discrete portion coincides with Klauder's coherent state (1) except for the normalization factor.

For the continuous spectrum (41) for $E \geqslant 0$, adopting the weighting function $\sigma(s) = \mathrm{e}^{-s}$ in (13), we take the limiting value,

$$\lim_{R\to\infty} [n]_R! = \rho(\varepsilon) = \int_0^\infty s^\varepsilon \mathrm{e}^{-s} \, \mathrm{d}s = \Gamma[\varepsilon+1] = \varepsilon!, \tag{52}$$

where $\varepsilon = E(k)/\omega = k^2/(2\omega)$. Note that writing $\Gamma(\varepsilon+1)$ formally as $\varepsilon!$ in (52) is to stress that it is a natural continuum limit of $[n]!$.

Then the continuous portion of the coherent states may be constructed in the form,

$$|s, \gamma\rangle_{\mathrm{cont}} = \mathcal{N}(s^2) \int_0^\infty \mathrm{d}\varepsilon \frac{s^\varepsilon \, \mathrm{e}^{-\mathrm{i}\gamma\varepsilon}}{\sqrt{\Gamma(\varepsilon+1)}} |\varepsilon\rangle_{\mathrm{cont}}, \tag{53}$$

expanded with the states $|\varepsilon\rangle$ satisfying

$$\hat{H}_\ell |\varepsilon\rangle = \omega\varepsilon|\varepsilon\rangle, \qquad \langle\varepsilon|\varepsilon'\rangle = \delta(\varepsilon - \varepsilon'). \tag{54}$$

The normalization factor $\mathcal{N}(s^2)$ common to the discrete and continuous portions is given by

$$\mathcal{N}(s^2)^{-2} = \sum_{n=0}^{\infty} \frac{s^{2n}}{[n]!} + \int_0^{\infty} d\varepsilon \, \frac{s^{2\varepsilon}}{\varepsilon!}, \tag{55}$$

which can be cast into the form,

$$\mathcal{N}(s^2)^{-2} = {}_2F_1(\ell + 2, \ell + 2; 2\ell + 3; (\ell + 1)^2 s^2) + \nu(s^2), \tag{56}$$

where $\nu(s)$ is the ν-function [13] defined by

$$\nu(x) = \int_0^{\infty} \frac{x^t}{\Gamma(t+1)} \, dt. \tag{57}$$

Consequently, as the flat-space limit of the coherent states (30) for the Coulomb problem in curved space, we obtain the coherent states for the radial Coulomb problem consisting of the discrete and continuous portions:

$$|s, \gamma\rangle = \mathcal{N}(s^2) \left[\sum_{n=0}^{\infty} \frac{s^n \, e^{-i\gamma[n]}}{\sqrt{[n]!}} |n\rangle_{\text{disc}} + \int_0^{\infty} d\varepsilon \, \frac{s^\varepsilon \, e^{-i\gamma\varepsilon}}{\sqrt{\varepsilon!}} |\varepsilon\rangle_{\text{cont}} \right]. \tag{58}$$

In particular, for the S-wave sector ($\ell = 0$), we have

$$|s, \gamma\rangle_{\ell=0} = \mathcal{N}_0(s^2) \left[\sum_{n=0}^{\infty} \frac{s^n \, e^{-i\gamma n(n+2)/(n+1)^2}}{\sqrt{(n+2)/(2n+2)}} |n, \ell = 0\rangle_{\text{disc}} + \int_0^{\infty} d\varepsilon \, \frac{s^\varepsilon \, e^{-i\gamma\varepsilon}}{\sqrt{\varepsilon!}} |\varepsilon, \ell = 0\rangle_{\text{cont}} \right], \tag{59}$$

with

$$\mathcal{N}_0(s^2)^{-2} = \frac{2}{s^2} \left[\frac{s^2}{1 - s^2} + \ln(1 - s^2) \right] + \nu(s^2). \tag{60}$$

The discrete portion of this S-wave limit coincides with the result given by Gazeau and Klauder [4].

The set of coherent states just obtained in the flat-space limit possesses all the properties (i)–(iii) posed by Klauder [1]. Naturally the resolution of unity is extended to include both the discrete and continuous states:

$$\int |s, \gamma\rangle\langle s, \gamma| \, d\mu(s, \gamma) = \mathbf{1}, \tag{61}$$

with the same measure as that of (6). This relation is not valid when the continuous states are ignored. In addition, the action identity of Gazeau and Klauder [4] is satisfied:

$$\langle s, \gamma | \hat{H}_\ell - E_0 | s, \gamma \rangle = \omega J \tag{62}$$

with the identification $J = s^2$.

4. Concluding remarks

We have obtained both the discrete and continuous portions of the coherent states for the radial Coulomb problem in a unified manner. We emphasize that the coherent states for the discrete spectrum in a uniformly curved space are reducible in the flat-space limit to those for the continuous spectrum plus those for the discrete spectrum. In the S-wave limit ($\ell = 0$), if the continuous part is ignored, our result (58) coincides with that of Gazeau

and Klauder [4] for the Coulomb-like discrete spectrum. Although the classical action-angle interpretation of $J = s^2$ and γ offered by Gazeau and Klauder is natural in (62) for the harmonic oscillator case, it is questionable that such an interpretation is appropriate for the Coulomb case. If $\langle J, \gamma | \hat{H} | J, \gamma \rangle$ corresponds to the classical energy E_{cl}, then $J = E_{cl}/\omega$ is the adiabatic invariant for the oscillator (after an appropriate adjustment of its dimension). The Bohr–Ishiwara–Sommerfeld–Wilson quantization $J \to n$ applied to the harmonic oscillator leads to $E_{cl} \to n\omega$, suggesting that the eigenvalues of $\hat{H}$ are $n\omega$. However, J does not seem to be an adiabatic invariant for any other systems in a strict sense. The semiclassical quantization $J \to n$ does not yield a correct spectrum for a system other than the harmonic oscillator. Nonetheless it is interesting to point out that the quantization condition, if modified as $J \to [n]$ (replacing the integer n by the generalized number $[n] = e_n$), leads to $E_{cl} \to e_n\omega$, and is compatible with the action-angle interpretation.

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